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ON PARTIALLY DEGENERATE BELL NUMBERS AND POLYNOMIALS

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Abstract. Recently, several authors have studied Bell numbers and polynomials, also called Tochard polynomials or exponential polynomials, by using and without using umbral calculus. In this paper, as a degenerate version of ordinary Bell numbers and polynomials, we introduce the partially degenerate Bell numbers and polynomials and give some new identities for those numbers and polynomials related with Stirling numbers of the first and second kind.

1. Introduction

As is well known, the Bell polynomials (also called Tochard polynomials or exponential polynomials and denoted by $\phi_n(x)$) are defined by the generating function

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}, \quad (\text{see } [14-20]). \tag{1.1}$$

The first few of them are

$$Bel_0(x) = 1,\ Bel_1(x) = x,\ Bel_2(x) = x^2 + x,\ Bel_3(x) = x^3 + 3x^2 + x,$$
$$Bel_4(x) = x^4 + 6x^3 + 7x^2 + x,\ Bel_5(x) = x^5 + 10x^4 + 25x^3 + 15x^2 + x,$$
$$Bel_6(x) = x^6 + 15x^5 + 65x^4 + 90x^3 + 35x^2 + x, \cdots,$$

From (1.1), we can easily derive the following equations:

$$Bel_n(x+y) = \sum_{l=0}^{n} \binom{n}{l} Bel_l(x)Bel_{n-l}(y),\ (n \geq 0), \quad (\text{see } [15,16]). \tag{1.2}$$

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For $n \geq 0$, the Stirling numbers of the first kind are defined as

$$(x)_0 = 1,\ (x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l,\ (n \geq 1). \tag{1.3}$$

and the Stirling numbers of the second kind are given by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad (\text{see } [1-13]). \tag{1.4}$$

It is well known that the generating function of $S_1(n,l)$ and $S_2(n,l)$ are given by

$$\Big(\log(1+t)\Big)^n = n! \sum_{m=n}^{\infty} S_1(m,n)\frac{t^m}{m!}, \tag{1.5}$$

and

$$(e^t-1)^n = n! \sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!}, \quad (\text{see } [12-13]). \tag{1.6}$$

From (1.1), we note that

$$\begin{aligned} e^{x(e^t-1)} &= \sum_{m=0}^{\infty} x^m \frac{1}{m!}(e^t-1)^m = \sum_{m=0}^{\infty} x^m \sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} x^m S_2(n,m) \right) \frac{t^n}{n!}. \end{aligned} \tag{1.7}$$

Thus by (1.1) and (1.7), we get

$$Bel_n(x) = \sum_{m=0}^{n} x^m S_2(n,m),\ (n \geq 0), \quad (\text{see } [14,15]). \tag{1.8}$$

When $x = 1$, $Bel_n = Bel_n(1) = \sum_{m=0}^{n} S_2(n,m)$ are called Bell numbers. Now, we observe that

$$\begin{aligned} e^{x(e^t-1)} &= e^{-x}e^{xe^t} = e^{-x}\sum_{l=0}^{\infty} \frac{x^l}{l!}e^{lt} \\ &= \sum_{n=0}^{\infty} \left(e^{-x} \sum_{l=0}^{\infty} \frac{x^l}{l!} l^n \right) \frac{t^n}{n!} \end{aligned} \tag{1.9}$$

From (1.1) and (1.9), we have

$$Bel_n(x) = e^{-x} \sum_{l=0}^{\infty} \frac{x^l}{l!} l^n, \quad (\text{see } [12,14,15]), \tag{1.10}$$

and

$$Bel_n = \frac{1}{e}\sum_{l=0}^{\infty}\frac{1}{l!}l^n, \ (n \geq 0). \tag{1.11}$$

Recently, the degenerate Bell polynomials were introduced by the generating function

$$(1+\lambda)^{\frac{x}{\lambda}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} = \sum_{n=0}^{\infty} Bel_{n,\lambda}(x)\frac{t^n}{n!}, \quad \text{(see [13])}. \tag{1.12}$$

When $x = 1$, $Bel_{n,\lambda} = Bel_{n,\lambda}(1)$ are called the degenerate Bell numbers. Note that $\lim_{\lambda\to 0} Bel_{n,\lambda}(x) = Bel_n(x)$, $(n \geq 0)$. Indeed, the degenerate Bell numbers and polynomials are given by

$$Bel_{n,\lambda} = \sum_{k=0}^{n}\sum_{m=0}^{k}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,k)S_2(k,m)\lambda^{n-k}, \tag{1.13}$$

and

$$Bel_{n,\lambda}(x) = \sum_{k=0}^{n}\sum_{m=0}^{k}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,k)S_2(k,m)\lambda^{n-k}x^m, \tag{1.14}$$

where $\lambda \in \mathbb{R}$ and $n \geq 0$, (see [13]). In this paper, as a degenerate version of ordinary Bell numbers and polynomials, we introduce the partially degenerate Bell numbers and polynomials and give some new identities for those numbers and polynomials related with Stirling numbers of the first and second kind.

2. Partially degenerate Bell numbers and polynomials

For $\lambda \in \mathbb{R}$, we consider the partially degenerate Bell polynomials which are defined by the generating function

$$F(t,x) = e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} = \sum_{n=0}^{\infty} bel_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.1}$$

When $x = 1$, $bel_{n,\lambda}(1) = bel_{n,\lambda}$ are called the partially degenerate Bell numbers. From (1.1) and (1.5), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}\lim_{\lambda\to 0} bel_{n,\lambda}(x)\frac{t^n}{n!} &= \lim_{\lambda\to 0} e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} \\ &= e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}.\end{aligned} \tag{2.2}$$

Thus by (2.2), we get

$$\lim_{\lambda\to 0} bel_{n,\lambda}(x) = Bel_n(x), \ (n \geq 0). \tag{2.3}$$

From (2.1), we can derive the following equations

$$\begin{aligned}\sum_{n=0}^{\infty} bel_{n,\lambda}(x)\frac{t^n}{n!} &= e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} = \sum_{m=0}^{\infty}\frac{x^m}{m!}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)^m \\ &= \sum_{m=0}^{\infty}\frac{x^m}{m!}\left(e^{\frac{1}{\lambda}\log(1+\lambda t)}-1\right)^m \\ &= \sum_{m=0}^{\infty} x^m \sum_{k=m}^{\infty} S_2(k,m)\lambda^{-k}\frac{1}{k!}\Big(\log(1+\lambda t)\Big)^k \\ &= \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} x^m S_2(k,m)\lambda^{-k}\right)\frac{1}{k!}\Big(\log(1+\lambda t)\Big)^k \\ &= \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} x^m S_2(k,m)\lambda^{-k}\right)\sum_{n=k}^{\infty} S_1(n,k)\frac{\lambda^n t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\sum_{m=0}^{k} S_2(k,m)S_1(n,k)\lambda^{n-k}x^m\right\}\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

By comparing the coefficients on both sides of (2.4), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$bel_{n,\lambda}(x) = \sum_{k=0}^{n}\sum_{m=0}^{k} S_2(k,m)S_1(n,k)\lambda^{n-k}x^m. \tag{2.5}$$

Note that

$$\lim_{\lambda\to 0} bel_{n,\lambda}(x) = \sum_{m=0}^{n} S_2(n,m)x^m = Bel_n(x), \ (n \geq 0).$$

From (2.1), we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} bel_{n,\lambda}(x)\frac{t^n}{n!} &= e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} = e^{-x}e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}\right)} \\ &= e^{-x}\sum_{k=0}^{\infty}\frac{x^k}{k!}(1+\lambda t)^{\frac{k}{\lambda}} \\ &= e^{-x}\sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{n=0}^{\infty}\left(\frac{k}{\lambda}\right)_n \lambda^n\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left\{e^{-x}\sum_{k=0}^{\infty}\frac{1}{k!}x^k(k|\lambda)_n\right\}\frac{t^n}{n!},\end{aligned} \tag{2.6}$$

where $(x|\lambda)_n = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$, $(n \geq 1)$ and $(x|\lambda)_0 = 1$. By (2.6), we get

$$bel_{n,\lambda}(x) = e^{-x}\sum_{k=0}^{\infty}\frac{1}{k!}(k|\lambda)_n x^k. \tag{2.7}$$

In particular, if we set $x = 1$, then we have

$$bel_{n,\lambda} = \frac{1}{e}\sum_{k=0}^{\infty}\frac{1}{k!}(k|\lambda)_n,\ (n \geq 0). \tag{2.8}$$

Therefore, by (2.7) and (2.8), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$bel_{n,\lambda}(x) = e^{-x}\sum_{k=0}^{\infty}\frac{1}{k!}(k|\lambda)_n x^k.$$

In particular,

$$bel_{n,\lambda} = \frac{1}{e}\sum_{k=0}^{\infty}\frac{1}{k!}(k|\lambda)_n.$$

From (2.1), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} bel_{n,\lambda}(x+y)\frac{t^n}{n!} &= e^{(x+y)\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\\ &= \left(e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\right)\left(e^{y\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\right)\\ &= \left(\sum_{l=0}^{\infty} bel_{l,\lambda}(x)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} bel_{m,\lambda}(y)\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} bel_{l,\lambda}(x) bel_{n-l,\lambda}(y)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.9}$$

Thus, by (2.9), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$bel_{n,\lambda}(x+y) = \sum_{l=0}^{n}\binom{n}{l} bel_{l,\lambda}(x) bel_{n-l,\lambda}(y).$$

Now, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} bel_{n,\lambda}(x)\frac{nt^{n-1}}{n!} &= \frac{\partial}{\partial t}F(t,x) = \frac{\partial}{\partial t}e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\\ &= x\frac{1}{1+\lambda t}(1+\lambda t)^{\frac{1}{\lambda}}e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}.\end{aligned} \tag{2.10}$$

It is not difficult to show that

$$\begin{aligned}(1+\lambda t)^{\frac{1}{\lambda}-1} &= \sum_{l=0}^{\infty}\left(\frac{1}{\lambda}-1\right)_l \lambda^l \frac{t^l}{l!} \\ &= \sum_{l=0}^{\infty}(1|\lambda)_{l+1}\frac{t^l}{l!}.\end{aligned} \tag{2.11}$$

From (2.10) and (2.11), we have

$$\begin{aligned}\sum_{n=1}^{\infty} bel_{n,\lambda}(x)\frac{t^{n-1}}{(n-1)!} &= \sum_{n=0}^{\infty} bel_{n+1,\lambda}(x)\frac{t^n}{n!} \\ &= x\left(\sum_{l=0}^{\infty}(1|\lambda)_{l+1}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} bel_{m,\lambda}(x)\frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(x\sum_{m=0}^{n}\binom{n}{m} bel_{m,\lambda}(x)(1|\lambda)_{n-m+1}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.12}$$

Therefore, by (2.12), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$bel_{n+1,\lambda}(x) = x\sum_{m=0}^{n}\binom{n}{m} bel_{m,\lambda}(x)(1|\lambda)_{n-m+1}.$$

In particular,

$$bel_{n+1,\lambda} = \sum_{m=0}^{n}\binom{n}{m} bel_{m,\lambda}(1|\lambda)_{n-m+1}.$$

For $n \in \mathbb{N}$, we have

$$\begin{aligned} bel_{n,\lambda}(x) &= e^{-x}\sum_{l=1}^{\infty}\frac{x^l}{l!}(l|\lambda)_n = e^{-x}x\sum_{l=0}^{\infty}\frac{x^l}{l!}\frac{(l+1|\lambda)_n}{l+1} \\ &= e^{-x}x\sum_{l=0}^{\infty}\frac{x^l}{l!}\frac{1}{l+1}\sum_{k=0}^{n}S_1(n,k)\lambda^{n-k}(l+1)^k \\ &= e^{-x}x\sum_{k=0}^{n}\sum_{l=0}^{\infty}\frac{x^l}{l!}\lambda^{n-k}S_1(n,k)(l+1)^{k-1} \\ &= e^{-x}x\sum_{k=0}^{n}\lambda^{n-k}S_1(n,k)\sum_{l=0}^{\infty}\frac{x^l}{l!}\sum_{j=0}^{k-1}\binom{k-1}{j}l^j \\ &= e^{-x}x\sum_{k=0}^{n}\lambda^{n-k}S_1(n,k)\sum_{l=0}^{\infty}\frac{x^l}{l!}\sum_{j=1}^{k}\binom{k-1}{j-1}l^{j-1} \\ &= x\sum_{k=1}^{n}\sum_{j=1}^{k}S_1(n,k)\lambda^{n-k}\binom{k-1}{j-1}e^{-x}\sum_{l=0}^{\infty}\frac{x^l}{l!}l^{j-1} \\ &= x\sum_{k=1}^{n}\sum_{j=1}^{k}S_1(n,k)\lambda^{n-k}\binom{k-1}{j-1}Bel_{j-1}(x). \end{aligned} \tag{2.13}$$

Therefore, we obtain the following theorem.

Theorem 2.5. *For $n \in \mathbb{N}$, we have*

$$bel_{n,\lambda}(x) = x\sum_{k=1}^{n}\sum_{j=1}^{k}\binom{k-1}{j-1}S_1(n,k)\lambda^{n-k}Bel_{j-1}(x).$$

From (2.1), we have

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{d}{dx}bel_{n,\lambda}(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\frac{d}{dx}bel_{n,\lambda}(x)\frac{t^n}{n!} = \frac{\partial}{\partial x}e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} \\ &= \left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} \\ &= (1+\lambda t)^{\frac{1}{\lambda}}e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} - e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}. \end{aligned} \tag{2.14}$$

Note that

$$(1+\lambda t)^{\frac{1}{\lambda}} = \sum_{l=0}^{\infty}(1|\lambda)_l\frac{t^l}{l!}. \tag{2.15}$$

By (2.14) and (2.15), we get

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{d}{dx}bel_{n,\lambda}(x)\frac{t^n}{n!} &= \left(\sum_{l=0}^{\infty}(1|\lambda)_l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}bel_{m,\lambda}(x)\frac{t^m}{m!}\right)-\sum_{n=0}^{\infty}bel_{n,\lambda}(x)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}bel_{m,\lambda}(x)(1|\lambda)_{n-m}-bel_{n,\lambda}(x)\right\}\frac{t^n}{n!}\\ &= \sum_{n=1}^{\infty}\left\{\sum_{m=0}^{n-1}\binom{n}{m}bel_{m,\lambda}(x)(1|\lambda)_{n-m}\right\}\frac{t^n}{n!}.\end{aligned} \tag{2.16}$$

Therefore, we obtain the following theorem.

Theorem 2.6. *For $n \geq 1$, we have*

$$\frac{d}{dx}bel_{n,\lambda}(x)=\sum_{m=0}^{n-1}\binom{n}{m}bel_{m,\lambda}(x)(1|\lambda)_{n-m}.$$

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